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Sensitivity Analysis of Nonnegative Irreducible Matrices

S. L. LIU AND S. Y. WANG
Institute of Systems Science, Chinese Academy of Sciences
Beijing 100080, P.R. China
<sliu><sywang>@iss02.iss.ac.cn

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Abstract—We develop a method based on the additive perturbation of a nonnegative irreducible matrix to analyze its sensitivity. Bounds for the norm of the difference between the perturbed right eigenvector and the initial one, and bounds for the difference between the perturbed principal eigenvalue and the initial one are obtained without any additional assumption on the nonnegative irreducible matrix.

Keywords—Nonnegative irreducible matrix, Sensitivity analysis, Principle eigenvalue, Principal eigenvector.

1. INTRODUCTION

Let A be a nonnegative irreducible $n \times n$ real matrix, λ_1 be the principle eigenvalue of A, and x_1 be the principal eigenvector corresponding to λ_1. An additive perturbation of A such as $A + \Delta A$ will produce perturbations in the principal eigenvalue $\lambda_1 + \Delta\lambda_1$ and the principal eigenvector $x_1 + \Delta x_1$. Under the assumption that A has elementary divisors, the changes in the principal eigenvalue and principal eigenvector are given by Faddeev and Fadeeva [1] as follows:

$$\Delta\lambda_1 \approx \frac{y_1 \Delta A x_1}{y_1^\top x_1}, \tag{1.1}$$

$$\Delta x_1 \approx \sum_{j=2}^{n} \left\{ \frac{y_j^T (\Delta A) x_1}{(\lambda_1 - \lambda_j)\, y_j^\top x_j} \right\} x_j, \tag{1.2}$$

respectively. The computation of Δx_1 requires the calculation of the eigenvalues λ_i, $i = 1, 2, \dots, n$, and the corresponding right and left eigenvectors x_i and y_i, $i = 1, 2, \dots, n$, respectively. This is a limitation for its applications.

Using the expression of x_1 given by Saaty [2] and Vargas [3] obtained the following result:

$$\Delta x_1 = \lim_{k \to \infty} \frac{k A^{k-1} \left[\|A^k\| (\Delta A) - \|A^{k-1}(\Delta A)\| A \right] e}{\|A^k\| \, \|k A^{k-1}(\Delta A) + A^k\|}, \tag{1.3}$$

where $\|A\| = e^\top A e$ and $e = (1, 1, \dots, 1)^\top$. But the sensitivity analysis he made was for a particular class of nonnegative irreducible matrices, called reciprocal matrices. Some other results on sensitivity analysis of reciprocal matrices can be found in [4–6].

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In this paper, we first give four lemmas. Based on these lemmas, we analyze sensitivity of a nonnegative irreducible matrix without any assumption. We obtain the bounds for the norm of the difference between the perturbed right eigenvector and the initial one, the bounds for the difference between the perturbed principal eigenvalue and the initial one, and the similar results for the left principal eigenvector.

2. NOTATIONS AND LEMMAS

Let A be an $n \times n$ matrix. Denote the i^{th} row and i^{th} column of A by $r_i(A)$ and $c_i(A)$, respectively, the rank of A by $R(A)$ and $\operatorname{diag}(x_1, x_2, \dots, x_n)$ by $[x]$, where $x = (x_1, x_2, \dots, x_n)^\top$. Denote the principal eigenvalue of a nonnegative irreducible matrix A by $\lambda(A)$.

LEMMA 2.1. *Let A be a nonnegative irreducible matrix. Let $x = (x_1, x_2, \dots, x_n)^\top$ and $y = (y_1, y_2, \dots, y_n)$ be, respectively, the right eigenvector and left eigenvector of A with respect to the principal eigenvalue $\lambda(A)$. Then*

$$r_i(A_1) = \lambda(A), \qquad i = 1, 2, \dots, n, \tag{2.1}$$

$$c_i(A_2) = \lambda(A), \qquad i = 1, 2, \dots, n, \tag{2.2}$$

where $A_1 = [x]^{-1}A[x]$ and $A_2 = [y]A[y]^{-1}$.

PROOF. Because $Ax = \lambda x$, we have that $\sum_{j=1}^n a_{ij}x_j = \lambda(A)x_i$ for $i = 1, 2, \dots, n$. By the Perron-Frobenius Theorem [7], $x_i > 0$, for $i = 1, 2, \dots, n$. Hence, $\sum_{j=1}^n a_{ij}x_j/x_i = \lambda(A)$. This is exactly the expression (2.1). We can similarly prove (2.2).

LEMMA 2.2. *(See [7].) For any $n \times n$ nonnegative irreducible matrix A,*

(i) $\min_i r_i(A) \le \lambda(A) \le \max_i r_i(A)$;
(ii) $\min_i c_i(A) \le \lambda(A) \le \max_i c_i(A)$.

LEMMA 2.3. *Suppose that $x = (x_1, x_2, \dots, x_n)^\top$ and $y = (y_1, y_2, \dots, y_n)$ are, respectively, the right eigenvector and left eigenvector of a nonnegative irreducible matrix A with respect to the principal eigenvalue $\lambda(A)$ satisfying $e^\top x = ye = 1$, where $e = (1, 1, \dots, 1)^\top$. Let R_i and C_i be the matrices obtained by replacing the i^{th} row and the i^{th} column of the matrix $\lambda(A)I - A$ by $e^\top$ and e, respectively. Then for $i = 1, 2, \dots, n$,*

(i) *R_i is reversible and $R_ix = (0, \dots, 1, \dots, 0)^\top$ in which the i^{th} element is 1;*
(ii) *C_i is reversible and $yC_i = (0, \dots, 1, \dots, 0)$ in which the i^{th} element is 1.*

PROOF. Because $(\lambda(A)I - A)x = 0$ and by the definition of R_i, we have that $R_ix = (0, \dots, 1, \dots, 0)^\top$. Now we show that R_i is reversible.

Since A is a nonnegative irreducible matrix, we have that $R(\lambda(A)I - A) = n - 1$ by the Perron-Frobenius Theorem. Let a_i denote the i^{th} row vector of $\lambda(A)I - A$. Since $y(\lambda(A)I - A) = 0$, by the Perron-Frobenius Theorem, $\sum_{i=1}^n y_i a_i = 0$, and $y_i > 0$ for all $i = 1, 2, \dots, n$. Hence, for any i, a_i can be linearly expressed by the other $a_j (j \ne i)$. Because $R(\lambda(A)I - A) = n - 1$, we know that any $n - 1$ row vectors of $\lambda(A)I - A$ are linearly independent. Suppose that

$$\sum_{j \ne i} \mu_j a_j + \mu_i e^\top = 0, \tag{2.3}$$

where μ_j is a real number, $j = 1, 2, \dots, n$. Multiplying the right side of (2.3) by the right principal eigenvector x of A and noting that $(\lambda(A)I - A)x = 0$ and $e^\top x = 1$, we get that $\mu_i = 0$. Substituting μ_i by 0 in (2.3), we have that $\mu_j = 0$ for all $j \ne i$. Thus, a set of row vectors $a_1, \dots, a_{i-1}, e, a_{i+1}, \dots, a_n$ of R_i are linearly independent. Therefore, R_i is reversible. The conclusion (ii) can be similarly proved.

The following lemma is obvious. To save space, we omit its proof.

LEMMA 2.4. *For any $n \times n$ matrix A, we have*

(i) $-\|A\|_\infty \le \min_i r_i(A) \le \max_i r_i(A) \le \|A\|_\infty$;

(ii) $-\|A\|_1 \le \min_i c_i(A) \le \max_i c_i(A) \le \|A\|_1$.

3. SENSITIVITY ANALYSIS OF A NONNEGATIVE IRREDUCIBLE MATRIX

In this section, we prove two sensitivity theorems.

THEOREM 3.1. *Let A be a nonnegative irreducible matrix and ΔA be its matrix of perturbations such that the matrix $A + \Delta A$ is also a nonnegative irreducible matrix. Let $x = (x_1, x_2, \dots, x_n)^\top$ and $y = (y_1, y_2, \dots, y_n)$ be the principal right and left eigenvectors of A, respectively. Then*

(i) $\min_i r_i(\Delta A_1) \le \lambda(\overline{A}) - \lambda(A) \le \max_i r_i(\Delta A_1)$;

(ii) $\min_i c_i(\Delta A_2) \le \lambda(\overline{A}) - \lambda(A) \le \max_i c_i(\Delta A_2)$,

where $\Delta A_1 = [x]^{-1}\Delta A[x]$ and $\Delta A_2 = [y]\Delta A[y]^{-1}$.

PROOF. Let $C = [x]^{-1}A[x]$ and $\overline{C} = [x]^{-1}\overline{A}[x]$. Thus, $\lambda(C) = \lambda(A)$, $\lambda(\overline{C}) = \lambda(\overline{A})$ and

$$\overline{C} = [x]^{-1}(A + \Delta A)[x] = C + \Delta C, \tag{3.1}$$

where $\Delta C = [x]^{-1}\Delta A[x]$. By (2.1) and (3.1), we have

$$r_i\left(\overline{C}\right) = \lambda(C) + r_i(\Delta C). \tag{3.2}$$

By Lemma 2.2 (i), we have

$$\min_i r_i\left(\overline{C}\right) \le \lambda\left(\overline{C}\right) \le \max_i r_i\left(\overline{C}\right). \tag{3.3}$$

Combining (3.2) and (3.3), we have

$$\min_i r_i(\Delta C) \le \lambda\left(\overline{C}\right) - \lambda(C) \le \max_i r_i(\Delta C). \tag{3.4}$$

Since $\lambda(C) = \lambda(A)$ and $\lambda(\overline{C}) = \lambda(\overline{A})$, we complete a proof for (i).

Similarly, we can prove (ii) by (2.2) and Lemma 2.2 (ii).

COROLLARY 3.2.

(i) $((\min_i x_i)/(\max_i x_i)) \min_i r_i(\Delta A) \le \lambda(\overline{A}) - \lambda(A) \le ((\max_i x_i)/(\min_i x_i)) \max_i r_i(\Delta A)$;

(ii) $((\min_j y_j)/(\max_j y_j)) \min_j c_j(\Delta A) \le \lambda(\overline{A}) - \lambda(A) \le ((\max_j y_j)/(\min_j y_j)) \max_j c_j(\Delta A)$;

(iii) $|\lambda(\overline{A}) - \lambda(A)| \le ((\max_i x_i)/(\min_i x_i))\|\Delta A\|_\infty$;

(iv) $|\lambda(\overline{A}) - \lambda(A)| \le ((\max_j y_j)/(\min_j y_j))\|\Delta A\|_1$.

PROOF. Let $\Delta A = (\Delta a_{ij})_{n\times n}$. Since $\Delta A_1 = [x]^{-1}\Delta A[x]$, $r_i(\Delta A_1) = \sum_{j=1}^n (x_j/x_i)\Delta a_{ij}$. Since all $x_i > 0$, we get (i). By Lemma 2.4 (i) and Theorem 3.1 (i), we have

$$\left|\lambda\left(\overline{A}\right) - \lambda(A)\right| \le \left\|[x]^{-1}\right\|_\infty \|[x]\|_\infty \|\Delta A\|_\infty = \frac{\max_i x_i}{\min_i x_i}\|\Delta A\|_\infty.$$

Thus, (iii) is true. Similarly, we can prove (ii) and (iv).

THEOREM 3.3. *Let A be a nonnegative irreducible matrix and ΔA be its matrix of perturbations such that $A + \Delta A$ is also a nonnegative irreducible matrix. Let $x = (x_1, x_2, \dots, x_n)^\top$ and $y = (y_1, y_2, \dots, y_n)$ be, respectively, the principal right and left eigenvectors of A, and $\overline{x}$ and $\overline{y}$ be, respectively, the principal right and left eigenvectors of $\overline{A}$, such that $e^\top x = e^\top \overline{x} = ye = \overline{y}e = 1$. Then,*

(i) $\|\overline{x} - x\|_1 \le K_{11}(\delta^* + \|\Delta A\|_1)$;

(ii) $\|\overline{x} - x\|_1 \le K_{11}(1 + (\max_j y_j)/(\min_j y_j))\|\Delta A\|_1$;
(iii) $\|\overline{x} - x\|_\infty \le K_{12}(\delta^* + \|\Delta A\|_\infty)$;
(iv) $\|\overline{x} - x\|_\infty \le K_{12}(1 + (\max_j x_j)/(\min_j x_j))\|\Delta A\|_\infty$;
(v) $\|(\overline{y} - y)^\top\|_1 \le K_{21}(\delta^* + \|\Delta A\|_\infty)$;
(vi) $\|(\overline{y} - y)^\top\|_1 \le K_{21}(1 + (\max_j x_j)/(\min_j x_j))\|\Delta A\|_\infty$;
(vii) $\|(\overline{y} - y)^\top\|_\infty \le K_{22}(\delta^* + \|\Delta A\|_1)$;
(viii) $\|(\overline{y} - y)^\top\|_\infty \le K_{22}(1 + (\max_j y_j)/(\min_j y_j))\|\Delta A\|_1$,

where $e = (1, 1, \dots, 1)^\top$, *and*

$$K_{11} = \min_i \left\|R_i^{-1}\right\|_1, \qquad K_{12} = \min_i \left\|R_i^{-1}\right\|_\infty, \qquad K_{21} = \min_i \left\|C_i^{-1}\right\|_\infty, \qquad K_{22} = \min_i \left\|C_i^{-1}\right\|_1,$$

$$\delta^* = \max\left\{\left|\max_i r_i(\Delta A_1)\right|, \left|\max_i c_i(\Delta A_2)\right|, \left|\min_i r_i(\Delta A_1)\right|, \left|\min_i c_i(\Delta A_2)\right|\right\}.$$

PROOF. Let $\overline{R}_i$ be the matrix obtained by replacing the i^{th} row vector of $\lambda(\overline{A})I - \overline{A}$ by $e^\top = (1, 1, \dots, 1)$. By Lemma 2.3 (i), $\overline{R}_i$ and R_i are reversible and satisfy

$$R_i x = \overline{R}_i \overline{x} = (0, \dots, 1, \dots, 0)^\top. \tag{3.5}$$

Let $F_i = \overline{R}_i - R_i$, i.e., F_i is the matrix obtained by replacing the i^{th} row of $(\lambda(\overline{A}) - \lambda(A))I - \Delta A$ by zero row. Thus, we have

$$\overline{R}_i \overline{x} - R_i x = (F_i + R_i)\overline{x} - R_i x = 0, \tag{3.6}$$

namely,

$$R_i(\overline{x} - x) = -F_i \overline{x}. \tag{3.7}$$

Since R_i is reversible, we have

$$\overline{x} - x = -R_i^{-1} F_i \overline{x}. \tag{3.8}$$

By Theorem 3.1 and Corollary 3.2 (iv), we have

$$\begin{aligned} \|F_i\|_1 &\le \left\|\left(\lambda\left(\overline{A}\right) - \lambda(A)\right) I - \Delta A\right\|_1 \\ &\le \delta^* + \|\Delta A\|_1, \end{aligned}$$

and

$$\begin{aligned} \|F_i\|_1 &\le \left\|\left(\lambda\left(\overline{A}\right) - \lambda(A)\right) I - \Delta A\right\|_1 \\ &\le \frac{\max_j y_j}{\min_j y_j}\|\Delta A\|_1 + \|\Delta A\|_1 \\ &= \left(1 + \frac{\max_j y_j}{\min_j y_j}\right)\|\Delta A\|_1. \end{aligned}$$

Noting that $\|\overline{x}\|_1 = 1$, we complete the proof of (i) and (ii) by taking column norm of (3.8).

Similarly, we can prove (iii)–(viii).

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